Numerical Methods
Using MATLAB
Third Edition

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Theorem 7.2 (Composite Trapezoidal Rule). Suppose that the interval \([a, b]\) is subdivided into \(M\) subintervals \([x_k, x_{k+1}]\) of width \(h = (b-a)/M\) by using the equally spaced nodes \(x_k = a + kh\), for \(k = 0, 1, \ldots, M\). The composite trapezoidal rule for \(M\) subintervals can be expressed in any of three equivalent ways:

\[
T(f, h) = \frac{h}{2} \sum_{k=1}^{M} (f(x_{k-1}) + f(x_k))
\]

or

\[
T(f, h) = \frac{h}{2} (f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{M-2} + 2f_{M-1} + f_M)
\]

or

\[
T(f, h) = \frac{h}{2} (f(a) + f(b)) + h \sum_{k=1}^{M-1} f(x_k).
\]

This is an approximation to the integral of \(f(x)\) over \([a, b]\), and we write

\[
\int_{a}^{b} f(x) \, dx \approx T(f, h).
\]

Proof. Apply the trapezoidal rule over each subinterval \([x_{k-1}, x_k]\) (see Figure 7.6). Use the additive property of the integral for subintervals:

\[
\int_{a}^{b} f(x) \, dx = \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} f(x) \, dx \approx \sum_{k=1}^{M} \frac{h}{2} (f(x_{k-1}) + f(x_k)).
\]

Since \(h/2\) is a constant, the distributive law of addition can be applied to obtain (1a). Formula (1b) is the expanded version of (1a). Formula (1c) shows how to group all the intermediate terms in (1b) that are multiplied by 2.

Corollary 7.2 (Trapezoidal Rule: Error Analysis). Suppose that \([a, b]\) is subdivided into \(M\) subintervals \([x_k, x_{k+1}]\) of width \(h = (b-a)/M\). The composite trapezoidal rule

\[
T(f, h) = \frac{h}{2} (f(a) + f(b)) + h \sum_{k=1}^{M-1} f(x_k)
\]

is an approximation to the integral

\[
\int_{a}^{b} f(x) \, dx = T(f, h) + E_T(f, h).
\]

Furthermore, if \(f \in C^2[a, b]\), there exists a value \(c\) with \(a < c < b\) so that the error term \(E_T(f, h)\) has the form

\[
E_T(f, h) = \frac{-(b-a)f^{(2)}(c)h^2}{12} = O(h^2).
\]
Theorem 7.3 (Composite Simpson Rule). Suppose that \([a, b]\) is subdivided into \(2M\) subintervals \([x_k, x_{k+1}]\) of equal width \(h = (b - a)/(2M)\) by using \(x_k = a + kh\) for \(k = 0, 1, \ldots, 2M\). The composite Simpson rule for \(2M\) subintervals can be expressed in any of three equivalent ways:

\[
S(f, h) = \frac{h}{3} \sum_{k=1}^{M} (f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})) \tag{4a}
\]

or

\[
S(f, h) = \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}) \tag{4b}
\]

or

\[
S(f, h) = \frac{h}{3} (f(a) + f(b)) + \frac{2h}{3} \sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3} \sum_{k=1}^{M} f(x_{2k-1}) \tag{4c}
\]

This is an approximation to the integral of \(f(x)\) over \([a, b]\), and we write

\[
\int_a^b f(x)\, dx \approx S(f, h). \tag{5}
\]

Proof. Apply Simpson's rule over each subinterval \([x_{2k-2}, x_{2k}]\) (see Figure 7.7). Use the additive property of the integral for subintervals:

\[
\int_a^b f(x)\, dx = \sum_{k=1}^{M} \int_{x_{2k-2}}^{x_{2k}} f(x)\, dx \tag{6}
\]

\[
\approx \sum_{k=1}^{M} \frac{h}{3} (f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})).
\]

Since \(h/3\) is a constant, the distributive law of addition can be applied to obtain (4a). Formula (4b) is the expanded version of (4a). Formula (4c) groups all the intermediate terms in (4b) that are multiplied by 2 and those that are multiplied by 4.

Corollary 7.3 (Simpson's Rule: Error Analysis). Suppose that \([a, b]\) is subdivided into \(2M\) subintervals \([x_k, x_{k+1}]\) of equal width \(h = (b - a)/(2M)\). The composite Simpson rule

\[
S(f, h) = \frac{h}{3} (f(a) + f(b)) + \frac{2h}{3} \sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3} \sum_{k=1}^{M} f(x_{2k-1}) \tag{14}
\]

is an approximation to the integral

\[
\int_a^b f(x)\, dx = S(f, h) + E_S(f, h). \tag{15}
\]

Furthermore, if \(f \in C^4[a, b]\), there exists a value \(c\) with \(a < c < b\) so that the error term \(E_S(f, h)\) has the form

\[
E_S(f, h) = \frac{-(b - a)f^{(4)}(c)h^4}{180} = O(h^4). \tag{16}
\]
## Composite Simpson's Rule

Table 7.3  The Composite Trapezoidal Rule for $f(x) = 2 + \sin(2\sqrt{x})$ over $[1, 6]$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$h$</th>
<th>$S(f, h)$</th>
<th>$E_S(f, h) = O(h^4)$</th>
</tr>
</thead>
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<tr>
<td>5</td>
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<td>0.000046371</td>
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<td>8.18344750</td>
<td>0.00003171</td>
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<tr>
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<td>8.18347717</td>
<td>0.00000204</td>
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<tr>
<td>40</td>
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<td>8.18347908</td>
<td>0.00000013</td>
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<td>80</td>
<td>0.03125</td>
<td>8.18347920</td>
<td>0.00000001</td>
</tr>
</tbody>
</table>

## Composite Trapezoidal Rule

Table 7.2  The Composite Trapezoidal Rule for $f(x) = 2 + \sin(2\sqrt{x})$ over $[1, 6]$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$h$</th>
<th>$T(f, h)$</th>
<th>$E_T(f, h) = O(h^2)$</th>
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<tr>
<td>10</td>
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