1. Let $X_1, X_2, \ldots$ be independent random variables with mean $\mu$ and variance $\sigma^2 < \infty$ and let $\bar{X}_n$ be the sample mean of the first $n$ observations. The Law of Large Numbers says that $\bar{X}_n \to_p \mu$. Write the mathematical expression implied by the statement that $\bar{X}_n \to_p \mu$? (Hint: The answer involves a limit.)

2. Let $X_1, \ldots, X_{25}$ be a random sample of beta random variables with shapes 2 and 8. Approximate $\Pr\left(\frac{1}{25} \sum_{i=1}^{25} X_i > 0.24\right)$ using $\Phi$, the cumulative distribution function of the standard normal distribution. Hint, the variance of a beta random variable with shapes $\alpha$ and $\beta$ is $\frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$. 

For all the remaining parts on this exam, suppose that \( X_1, \ldots, X_n \) are independent, where \( X_i \) given \( \lambda \) has a Poisson distribution with rate \( \lambda \). The probability mass function of \( X_i \) given \( \lambda \) is:

\[
p(x_i | \lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \quad \text{for } x_i \in \{0, 1, \ldots\}.
\]

Let the classical estimator of \( \lambda \) be:

\[
\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i
\]

Assume a gamma prior on \( \lambda \). (Hint: Your calculations will be easier if you use parameterize the gamma distribution in terms of a shape and rate.) For Bayesian estimation, use squared error loss.

3. Succinctly write the model outlined above in distributional notation (e.g., \( X \sim \text{Normal}(\mu, \sigma) \)). Name the two parts of the model. Note that this part does not involve any densities.

4. Write \( p(x_1, \ldots, x_n | \lambda) \), the joint density of the sampling model for \( X_1, \ldots, X_n \). A random sample justifies what key assumption used to find this joint density?

5. For this item only, assume that \( n = 3, x_1 = 3, x_2 = 5, \) and \( x_3 = 4 \). Write the likelihood function for \( \lambda \) as simply as possible.
6. Write \( p(\lambda) \), the density of the prior distribution for \( \lambda \).

7. Find \( p(\lambda \mid x_1, \ldots, x_n) \), the density of the posterior distribution of \( \lambda \).

8. What is the posterior distribution of \( \lambda \)? That is, what is the conditional distribution of \( \lambda \) given \( X_1, \ldots, X_n \)? Note that this part does not ask for a density.
9. What is the Bayes estimator of $\lambda$?

10. Show that the Bayes estimator of $\lambda$ is a weighted average of the prior estimate of $\lambda$ and the classical estimator of $\lambda$.

11. Find the bias of the classical estimator of $\lambda$.

12. Find the bias of the Bayes estimator of $\lambda$. 
13. Find the mean squared error (MSE) of the classical estimator of $\lambda$.

14. Find the mean squared error (MSE) of the Bayes estimator of $\lambda$. 
15. What values for the shape and rate parameters in the gamma prior are compatible with prior estimate for $\lambda$ being 4 and effective number of observations from the prior being 10?

16. Using values from the previous problem and assuming $n = 10$, give a value for $\lambda$ for which the Bayes estimator is superior in terms of mean squared error. Likewise, give a value for $\lambda$ for which the Bayes estimator is inferior in terms of mean squared error. Justify your choices mathematically.